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ON STABILITY OF THE EQUILIBRIUM POSITION OF VIBRATIONAL SYSTEMS WITH VARIABLE COEFFICIENTS*

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Sufficient conditions are obtained for the asymptotic stability of the equilibrium position of a one-dimensional linear non-autonomous vibrational system. It is shown that these conditions are close to necessary and sufficient conditions in a known sense.

Let us consider a pendulum described by the differential equation

$$x^{\cdot\cdot} + f(t)\,x^{\cdot} + g(t)\,x = 0 \tag{1}$$

where $f(t)$ and $g(t)$ are known continuous bounded functions of time. By using the Liapunov function taken in the form of a quadratic form with constant coefficients relative to $x, x^{\cdot}$, a sufficient condition for the asymptotic stability is obtained in /1/ (under the assumption that $0 < l \leqslant f(t) \leqslant L,\ 0 < m \leqslant g(t) \leqslant M$)

$$x = 0,\ x^{\cdot} = 0 \tag{2}$$

for the solution of (1) which took the form of constraints on the constants l, L, m, M. Let us note that the question of the asymptotic stability of the trivial solution was considered by many authors /2—5/ for the general case of an n-dimensional linear system $x^{\cdot} = A(t)x$; however, the conditions obtained impose sufficiently strict constraints on the elements of the matrix $A(t)$.

Let us assume the function $g(t)$ to be continuously differentiable and the inequalities

$$|f(t)| < M_1,\ |g(t)| < M_2,\ |g^{\cdot}(t)| < M_3$$

to be valid.

Theorem. If the following conditions are satisfied

$$g(t) > \alpha_1^2 > 0,\quad p(t) = \frac{1}{2}\frac{g^{\cdot}(t)}{g(t)} + f(t) > \alpha_2^2 > 0 \tag{3}$$

then the solution (2) of the differential equation (1) is uniformly asymptotically stable in the initial time t_0 and the initial perturbations $x(t_0), x^{\cdot}(t_0)$.

Proof. Let us consider the function

$$V_1 = \frac{1}{2}\left(x^2 + 2\beta\frac{xx^{\cdot}}{\sqrt{g(t)}} + \frac{x^{\cdot 2}}{g(t)}\right) \quad (\beta = \text{const})$$

By virtue of (1) its derivative has the form

$$V_1^{\cdot} = \frac{1}{\sqrt{g(t)}}\left[\left(-\frac{p(t)}{\sqrt{g(t)}} + \beta\right)x^{\cdot 2} - \beta p(t)\,xx^{\cdot} - \beta g(t)\,x^2\right]$$

If the quantity $\beta > 0$ is taken sufficiently small, then V_1 will be positive-definite $(V_1 > 0)$ and $V_1^{\cdot}$ will be negative-definite. It can be shown that for the satisfaction of the conditions $V_1 > 0,\ V_1^{\cdot} < 0$, for example,

$$0 < \beta < \min\left\{1,\ \frac{\alpha_2^2}{2\sqrt{M_2}},\ \frac{8\alpha_1^6\alpha_2^2}{(M_3 + 2\alpha_1^2 M_1)^2\sqrt{M_2}}\right\}$$

are sufficient.

Therefore, all the conditions are satisfied for the Liapunov theorem on asymptotic stability /6/, therefore, the unperturbed motion (2) is uniformly asymptotically stable in the initial time t_0 and the initial perturbations $x(t_0), x^{\cdot}(t_0)$, Q.E.D.

Corollary. Upon compliance with conditions (3) there exist positive numbers B, α such that for $t \geqslant t_0$ the following inequalities are valid

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$$|x(t)| < B \exp[-\alpha(t-t_0)], \quad |x^{\cdot}(t)| < B \exp[-\alpha(t-t_0)]$$

The proof results from the theorem presented and the results in /7/.

Remarks. 1°. The inequalities (3) that characterize the generalized positivity of the functions $g(t)$ and $p(t)$ are sufficient conditions for the asymptotic stability of the solution (2) of equation (1), which are close to necessary and sufficient conditions in the following sense. If the generalized positivity of either of the functions $g(t)$ and $p(t)$ or of both simultaneously is replaced by their generalized negativity in the inequalities (3), then the unperturbed motion (2) is unstable.

In fact, for instance, let the inequalities $p(t) < -\alpha_1^2 < 0$, $g(t) > \alpha_2^2 > 0$ be satisfied. We take the negative number β so small in absolute value that $V_1 > 0$, $V_1^{\cdot} > 0$. It hence follows that under the given assumptions the equilibrium position is unstable. If, however, one of the conditions

$$p(t) > \alpha_1^2 > 0, \quad g(t) < -\alpha_2^2 < 0 \tag{4}$$

$$p(t) < -\alpha_1^2 < 0, \quad g(t) < -\alpha_2^2 < 0 \tag{5}$$

is valid, then we take as Liapunov function

$$V_2 = \frac{1}{2}\left[x^2 + 2\beta \frac{xx^{\cdot}}{\sqrt{-g(t)}} + \frac{x^{\cdot 2}}{g(t)}\right]$$

By virtue of (1) its derivative will have the form

$$V_2^{\cdot} = \frac{1}{\sqrt{-g(t)}}\left\{\left[\frac{p(t)}{\sqrt{-g(t)}} + \beta\right] x^{\cdot 2} - \beta p(t)\, x x^{\cdot} - \beta g(t)\, x^2\right\}$$

By selecting the real constant β sufficiently small in absolute value, we can make the function $V_2^{\cdot}$ sign-definite (where we put $\beta > 0$ in the case (4), and $\beta < 0$) in the case (5)). Taking into account that V_2 is sign-variable, we conclude that the solution (2) of equation (1) is also unstable in the cases (4) and (5).

2°. If $f(t)$ and $g(t)$ are constants, then conditions (3) go over into the ordinary Routh–Hurwitz conditions.

REFERENCES

1. STARZHINSKII V.M., Sufficient conditions for the stability of a mechanical system with one degree of freedom, PMM, Vol.16, No.3, 1952.
2. LEBEDEV A.A., On a method of constructing Liapunov's functions, PMM, Vol.21, No.1, 1957.
3. RAZUMIKHIN B.S., Estimates of the solutions of a system of differential equations of perturbed motion with variable coefficients, PMM, Vol.21, No.1, 1957.
4. BOYKO A.G., On asymptotic stability conditions for the zero-th solution of a system of ordinary linear differential equations with variable coefficients, In: Asymptotic Methods in Nonlinear Mechanics. Inst. of Math. Acad. of Sciences. UkrSSR, Kiev, 1974.
5. LIASHCHENKO N.Ia., On the asymptotic stability of solutions of systems of differential equations. Dokl. Akad. Nauk SSSR, Vol.96, No.2, 1954.
6. KRASOVSKII N.N., Certain Problems of the Theory of Stability of Motion. FIZMATGIZ, Moscow, 1959. (See also, in English, Stability of Motion, Stanford Univ. Press, Stanford, Cal. 1963).
7. PERSIDSKII K.P., On the theory of stability of integrals of systems of differential equations. Izv. Fiz.-Matem. Ob-va i Nauchno-Issled. In-ta Matem. i Mekhan., Kazansk. Un-ta, Ser. 3, Vol.8, 1936-1937.

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